

# THE REARRANGEMENT-INVARIANT SPACE $\Gamma_{p,\phi}$

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ABSTRACT. Fix  $b \in (0, \infty)$  and  $p \in (1, \infty)$ . Let  $\phi$  be a positive measurable function on  $I_b := (0, b)$ . Define the Lorentz Gamma norm,  $\rho_{p,\phi}$ , at the measurable function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  by  $\rho_{p,\phi}(f) := \left[ \int_0^b f^{**}(t)^p \phi(t) dt \right]^{\frac{1}{p}}$ , in which  $f^{**}(t) := t^{-1} \int_0^t f^*(s) ds$ , where  $f^*(t) := \mu_f^{-1}(t)$ , with  $\mu_f(s) := |\{x \in I_b : |f(x)| > s\}|$ .

Our aim in this paper is to study the rearrangement-invariant space determined by  $\rho_{p,\phi}$ . In particular, we determine its Köthe dual and its Boyd indices. Using the latter a sufficient condition is given for a Caldéron-Zygmund operator to map such a space into itself.

## 1. INTRODUCTION

Let  $(X, \mu)$  be a  $\sigma$ -finite measure space with  $\mu(X) = b$  and denote by  $\mathfrak{M}(X)$  the set of  $\mu$ -measurable real-valued functions on  $X$ . This paper is concerned with the properties of certain rearrangement invariant spaces of functions in  $\mathfrak{M}(X)$ . The norm of such a space is defined in terms of an index  $p$ ,  $1 < p < \infty$ , and a positive locally integrable (weight) function  $\phi$  on  $I_b := (0, b)$  by

$$(1.1) \quad \rho_{p,\phi}(f) := \left[ \int_0^b f^{**}(t)^p \phi(t) dt \right]^{\frac{1}{p}}, \quad f \in \mathfrak{M}(X).$$

Here,

$$f^{**}(t) := t^{-1} \int_0^b f^*(s) ds, \quad t \in I_b,$$

in which the decreasing rearrangement,  $f^*$ , is the inverse (in a generalized sense) of the distribution function,  $\mu_f$ , of  $f$ , where

$$\mu_f(\lambda) := \mu(\{x \in X : |f(x)| > \lambda\}), \quad \lambda > 0.$$

We require

$$\int_1^\infty \phi(t) t^{-p} dt < \infty, \text{ if } b = \infty, \text{ and } \int_{I_b} \phi(t) t^{-p} dt = \infty, \text{ for all } b \in \mathbb{R}_+;$$

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2000 *Mathematics Subject Classification.* Primary 46E30; Secondary 26D10.

*Key words and phrases.* Lorentz Gamma space, Köthe dual space, weighted norm inequalities, Hardy operator, Stieltjes transform, Caldéron-Zygmund operator.

The research of the first author was partially supported by the grant no. 201/08/0383 of the Grant Agency of the Czech Republic and RVO: 67985840.

The research of the second author was supported in part by NSERC grant A4021.

otherwise, the space

$$\Gamma_{p,\phi} = \Gamma_{p,\phi}(X) := \{f \in \mathfrak{M}(X) : \rho_{p,\phi}(f) < \infty\}$$

would, in the first case, consist only of the zero function and, in the second case, would be equal to the space  $L_1(X)$  of  $\mu$ -integrable functions on  $X$ . Such weights  $\phi$  will be called non-trivial.

The spaces  $\Gamma_{p,\phi}$  are examples of rearrangement-invariant (r.i) Banach function spaces, which are defined by norms  $\rho$  whose characteristic property is that  $\rho(f) = \rho(g)$  whenever  $f, g \in \mathfrak{M}(X)$  are equimeasurable in the sense that  $f^* = g^*$ .

A key thing to know about a Banach function norm,  $\rho$ , such as (1.1), is its associate norm,  $\rho'$ , defined at  $g \in \mathfrak{M}(X)$  by

$$\rho'(g) = \sup_{\substack{f \in \mathfrak{M}(X) \\ \rho(f) \leq 1}} \int_X |fg| d\mu.$$

We will show that, when  $\Gamma_{p,\phi}(X) \not\supset L_\infty(X)$ , or, equivalently,  $\int_{I_b} \phi(s) ds = \infty$ , one has

$$\rho'_{p,\phi}(g) \approx \rho_{p',\psi}(g), \quad g \in \mathfrak{M}(X),$$

where  $p' = \frac{p}{p-1}$  and  $\psi$  is a certain (dual) weight.

We motivate the choice of  $\psi$ , in an appendix to the paper. For now, we just state our main result, namely,

**Theorem A** *Let  $(X, \mu)$  be a  $\sigma$ -finite measure space with  $\mu(X) = b$ . Fix  $p$ ,  $1 < p < \infty$ , and suppose  $\phi$  is a non-trivial weight function on  $I_b$ . Then,*

$$\rho'_{p,\phi}(g) \approx \rho_{p',\psi}(g) + \frac{\int_X |g|}{\left[ \int_{I_b} \phi \right]^p}, \quad g \in \mathfrak{M}(X),$$

in which

$$\psi(t) := \frac{t^{p+p'-1} \int_0^t \phi \int_t^b \phi(s) s^{-p} ds}{\left[ \int_0^t \phi + t^p \int_t^b \phi(s) s^{-p} ds \right]^{p'+1}}, \quad t \in I_b, \quad p' = \frac{p}{p-1}.$$

A proof of this theorem has been given by the first author and L. Pick in [3] using so-called discretization methods. Our aim here is to give a new proof using more familiar techniques. Alternative descriptions of the function space dual to  $\Gamma_{p,\phi}$  can be found in [4] and [8].

The Boyd indices of an r.i. norm are essential to describing the action of such operators as those of Calderón-Zygmund on the space  $L_\rho(\mathbb{R}^n)$ . These indices are defined in terms of the norm,  $h_\rho(s)$ , of the dilation operator. Their calculation when  $\rho = \rho_{p,\phi}$  and  $\mu(X) = \infty$  is greatly simplified by the result in

**Theorem B** Fix an index  $p$ ,  $1 < p < \infty$  and let  $\phi$  be a non-trivial weight on  $\mathbb{R}_+$ . Take  $\rho = \rho_{p,\phi}$  and at  $s \in \mathbb{R}_+$  set

$$h_\rho(s) := \sup \frac{\rho\left(f\left(\frac{t}{s}\right)\right)}{\rho(f)} = \sup \frac{\rho\left(f^*\left(\frac{t}{s}\right)\right)}{\rho(f^*)}, \quad 0 \neq f \in \mathfrak{M}_+(\mathbb{R}_+).$$

Then,

$$h_\rho(s) \approx \sup_{t \in \mathbb{R}_+} \left[ \frac{\int_0^{st} \phi(y) dy + s^p t^p \int_{st}^b \phi(y) y^{-p} dy}{\int_0^t \phi(y) dy + t^p \int_t^b \phi(y) y^{-p} dy} \right]^{\frac{1}{p}}.$$

## 2. REARRANGEMENT-INVARIANT SPACES

Let  $(X, \mu)$  be a  $\sigma$ -finite measure space with  $\mu(X) = b$  and denote by  $\mathfrak{M}(X)$  the set of  $\mu$ -measurable real-valued functions on  $X$  and by  $\mathfrak{M}_+(X)$  the nonnegative functions in  $\mathfrak{M}(X)$ . A Banach function norm is a functional  $\rho : \mathfrak{M}_+(X) \rightarrow \mathbb{R}_+$  satisfying

- (A1)  $\rho(f) = 0$  if and only if  $f = 0$   $\mu$ -a.e.,
- (A2)  $\rho(cf) = c\rho(f)$ ,  $c \geq 0$ ,
- (A3)  $\rho(f + g) \leq \rho(f) + \rho(g)$ ,
- (A4)  $0 \leq f_n \uparrow f$  implies  $\rho(f_n) \uparrow \rho(f)$ ,
- (A5)  $|E| < \infty$  implies  $\rho(\chi_E) < \infty$ ,
- (A6)  $|E| < \infty$  implies  $\int_E f d\mu \leq c_E(\rho)\rho(f)$ , for some constant  $c_E(\rho)$  depending on  $E$  and  $\rho$  but not on  $f \in \mathfrak{M}_+(X)$ .

Furthermore, as mentioned in the introduction, a Banach function norm is said to be rearrangement invariant if  $\rho(f) = \rho(g)$  whenever  $f, g \in \mathfrak{M}_+(X)$  are equimeasurable in the sense that  $f^* = g^*$ . The decreasing rearrangement,  $f^*$ , of  $f \in \mathfrak{M}(X)$  on  $\mathbb{R}_+$  is defined as

$$f^*(t) := \inf\{\lambda > 0 : \mu(\{x \in X : |f(x)| > \lambda\}) \leq t\},$$

$t \in I_b$ . It satisfies the property that

$$|\{t \in I_b : f^*(t) > \tau\}| = \mu(\{x \in X : |f(x)| > \tau\}), \quad f \in \mathfrak{M}(X), \quad \tau \in \mathbb{R}_+.$$

Now, although the mapping  $f \mapsto f^*$  is not subadditive, the mapping  $f \mapsto t^{-1} \int_0^t f^*(s) ds$  is, namely ,

$$(2.1) \quad t^{-1} \int_0^t (f + g)^*(s) ds \leq t^{-1} \int_0^t f^*(s) ds + t^{-1} \int_0^t g^*(s) ds,$$

for all  $f, g \in \mathfrak{M}(X)$ ,  $t \in I_b$ . The Kothe dual of a Banach function norm  $\rho$  is another such norm,  $\rho'$ , with

$$(2.2) \quad \rho'(g) := \sup_{\rho(f) \leq 1} \int_X fg d\mu, \quad f, g \in \mathfrak{M}_+(X).$$

It obeys the Principle of Duality; that is,

$$\rho'' := (\rho')' = \rho.$$

The space  $L_\rho(X)$  is the vector space

$$\{f \in \mathfrak{M}(X) : \rho(|f|) < \infty\},$$

together with the norm

$$\|f\|_{L_\rho} := \rho(|f|).$$

This Banach space is said to be an r.i. space provided  $\rho$  is an r.i. function norm. The norm,  $\rho_{p,\phi}$ , defined in (1.1) in terms of an index  $p$ ,  $1 < p < \infty$ , and a positive locally integrable (weight) function  $\phi$  on  $I_b$  is an r.i. norm;

If  $\rho$  is an r.i. function norm, then,

$$(2.3) \quad \rho(\chi_{(0,t)}) = \frac{t}{\rho'(\chi_{(0,t)})}, \quad t \in I_b$$

The dilation operator,  $E_s$ ,  $s \in \mathbb{R}_+$ , given at  $f \in \mathfrak{M}(\mathbb{R}_+)$ ,  $t \in \mathbb{R}_+$ , by

$$(E_s f)(t) := f(st),$$

is bounded on any r.i. space  $L_\rho(\mathbb{R}_+)$  and the operator norm of  $E_{1/s}$  on  $L_\rho(\mathbb{R}_+)$  is denoted by  $h_\rho(s)$ . The norm is determined on the non-negative decreasing functions in  $L_\rho(\mathbb{R}_+)$ .

We define the lower and upper Boyd indices of  $L_\rho(\mathbb{R}_+)$  as

$$i_\rho := \sup_{0 < t < 1} \frac{\log h_\rho(t)}{\log t} \quad \text{and} \quad I_\rho := \inf_{1 < t < \infty} \frac{\log h_\rho(t)}{\log t}$$

The operator norm of  $E_{1/s}$  on characteristic functions of the form  $\chi_{(0,a)}$ ,  $a \in \mathbb{R}_+$ , is denoted by  $M_\rho(s)$ ; thus,

$$M_\rho(s) = \sup_{0 < a < \infty} \frac{\rho(\chi_{(0,as)})}{\rho(\chi_{(0,a)})}.$$

The so-called fundamental indices of  $\rho$  are defined in terms of  $M_\rho$  as

$$\underline{i}_\rho := \sup_{0 < s < 1} \frac{\log M_\rho(s)}{\log s} \quad \text{and} \quad \underline{I}_\rho := \inf_{1 < s < \infty} \frac{\log M_\rho(s)}{\log s}.$$

Clearly,

$$0 \leq i_\rho \leq \underline{i}_\rho \leq \underline{I}_\rho \leq I_\rho \leq 1.$$

### 3. WEIGHTED SPACES

Fix  $b > 0$  and let  $w \in \mathfrak{M}_+(I_b)$ ,  $w > 0$  a.e.. Given  $p$ ,  $1 < p < \infty$ , the weighted Lebesgue space,  $L_p(w)$ , is defined by the norm

$$\left[ \int_0^b |f(t)|^p w(t) dt \right]^{\frac{1}{p}}, \quad f \in \mathfrak{M}(I_b).$$

One readily shows that the Banach dual of  $L_p(w)$  is the space  $L_{p'}(w^{1-p'})$ ,  $p' = \frac{p}{p-1}$ , namely, the weighted Lebesgue space with norm

$$\left[ \int_0^b |g(t)|^{p'} w(t)^{1-p'} dt \right]^{\frac{1}{p'}}, \quad g \in \mathfrak{M}(I_b).$$

In this section we consider the action of certain positive integral operators on such spaces. This action is expressed on terms of so-called weighted norm inequalities. The most basic ones involve the Hardy averaging operator and its dual, that is,

$$(Pf)(t) := t^{-1} \int_0^t f(s) ds \quad \text{and} \quad (Qf)(t) := \int_t^b f(s) \frac{ds}{s}, \quad f \in \mathfrak{M}_+(I_b), \quad t \in I_b.$$

**Theorem 3.1** ([6]). *Fix  $b > 0$  and let  $u$  and  $v$  be weights on  $I_b$ . Then, for  $1 < p \leq q < \infty$  one has the least constant  $C > 0$  in the inequality*

$$(3.1) \quad \left( \int_0^b (u(t)(Pf)(t))^q dt \right)^{\frac{1}{q}} \leq C \left( \int_0^b (v(t)f(t))^p dt \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}_+(I_b),$$

equivalent to

$$\sup_{0 < r < b} \left( \int_r^b \left( \frac{u(t)}{t} \right)^q dt \right)^{\frac{1}{q}} \left( \int_0^r v(t)^{-p'} dt \right)^{\frac{1}{p'}},$$

and the least constant  $C > 0$  in the inequality

$$(3.2) \quad \left( \int_0^b (u(t)(Qf)(t))^q dt \right)^{\frac{1}{q}} \leq C \left( \int_0^b (v(t)f(t))^p dt \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}_+(I_b),$$

equivalent to

$$\sup_{0 < r < b} \left( \int_0^r u(t)^q dt \right)^{\frac{1}{q}} \left( \int_r^b (tv(t))^{-p'} dt \right)^{\frac{1}{p'}}.$$

An operator essentially built from  $P$  and  $Q$  when  $b = \infty$  is the Stieltjes operator

$$(Sf)(t) := \int_0^\infty \frac{f(s)}{s+t} ds, \quad f \in \mathfrak{M}_+(I_b).$$

Clearly, for  $f \in \mathfrak{M}_+(\mathbb{R}_+)$ ,  $t \in \mathbb{R}_+$ ,

$$(3.3) \quad \frac{1}{2}[(Pf)(t) + (Qf)(t)] \leq (Sf)(t) \leq [(Pf)(t) + (Qf)(t)].$$

The following results are given in Andersen [1] for  $1 < p \leq q < \infty$  and in Sinnamon [7] for  $1 < q < p < \infty$ .

**Theorem 3.2.** *Let  $u$  and  $v$  be weights on  $\mathbb{R}_+$ . Then, in the inequality*

$$(3.4) \quad \left( \int_0^\infty (Sf)(t)^q u(t) dt \right)^{\frac{1}{q}} \leq K \left( \int_0^\infty f(t)^p v(t) dt \right)^{\frac{1}{p}}, \quad f \in \mathfrak{M}_+(\mathbb{R}_+),$$

the least possible  $K > 0$  is equivalent to

$$(3.5) \quad \sup_{t>0} \left( \int_0^\infty \left( \frac{t}{s+t} \right)^q u(s) ds \right)^{\frac{1}{q}} \left( \int_0^\infty \frac{v(t)^{1-p'}}{(s+t)^{p'}} ds \right)^{\frac{1}{p'}},$$

when  $1 < p \leq q < \infty$ , and to

$$\left[ \int_0^\infty \left[ \left[ \int_0^\infty \left( \frac{t}{s+t} \right)^q u(s) ds \right]^{\frac{1}{p}} \left[ \int_0^\infty \frac{v(t)^{1-p'}}{(s+t)^{p'}} ds \right]^{\frac{1}{p'}} \right]^{\frac{pq}{p-q}} u(t) dt \right]^{\frac{1}{q} - \frac{1}{p}},$$

when  $1 < q < p < \infty$ .

#### 4. PROOF OF THEOREM A.

The following lemma is a key element in the proof of the Theorem A. In it and in the rest of the section, it will simplify things if we write  $\psi$  in the form

$$(4.1) \quad \psi = \frac{(P\phi)(Q_p\phi)}{[(PQ_p)(\phi)]^{p'+1}},$$

where

$$(P\phi)(t) = t^{-1} \int_0^t \phi(s) ds \quad \text{and} \quad (Q_p\phi)(t) = pt^{p-1} \int_t^b \phi(s) s^{-p} ds.$$

**Lemma 4.1.** Fix  $p$  and  $b$  with,  $1 < p < \infty$  and  $0 < b \leq \infty$ . Suppose  $\phi$  is a non-trivial weight on  $I_b$  and let  $\psi$  be given by (4.1). Then, there exists  $C > 0$ , independent of  $f, g \in \mathfrak{M}_+(I_b)$ , such that

$$(i) \quad \int_{I_b} fg \left[ \frac{P\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p'}+1} \leq C \left( \int_{I_b} f^p \phi \right)^{\frac{1}{p}} \left[ \left( \int_{I_b} (Pg)^{p'} \psi \right)^{\frac{1}{p'}} + \frac{\int_{I_b} g}{\left[ \int_{I_b} \phi \right]^{\frac{1}{p}}} \right],$$

if  $f \downarrow$ , and

$$(ii) \quad \int_{I_b} fg \left[ \frac{Q_p\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p'}+1} \leq C \left( \int_{I_b} f(t)^p \phi(t) t^{-p} dt \right)^{\frac{1}{p}} \left( \int_{I_b} \left( \int_t^b g \right)^{p'} \psi(t) dt \right)^{\frac{1}{p'}},$$

if  $f \uparrow$ .

*Proof.* (i) We have

$$\begin{aligned} & \frac{p'}{p'+1} \int_{I_b} fg \left[ \frac{P\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p'}+1} \\ &= \int_{I_b} g(t) \int_0^t f(s) \left( \int_0^s \phi \right)^{\frac{1}{p'}} \phi(s) ds [t(PQ_p)(\phi)(t)]^{-\frac{1}{p'}-1} dt, \quad \text{since } f \downarrow, \end{aligned}$$

$$\begin{aligned}
&= \int_{I_b} f(t) \left( \int_0^t \phi \right)^{\frac{1}{p'}} \int_t^b g(s) [s (PQ_p)(\phi)(s)]^{-\frac{1}{p'}-1} ds \phi(t) dt, \quad \text{by Fubini's theorem,} \\
&= \int_{I_b} f(t) \left( \int_0^t \phi \right)^{\frac{1}{p'}} \left[ \int_0^s g [s (PQ_p)(\phi)(s)]^{-\frac{1}{p'}-1} \Big|_t^b \right. \\
&\quad \left. + \left( \frac{1}{p'} + 1 \right) \int_t^b \int_0^s g [s (PQ_p)(\phi)(s)]^{-\frac{1}{p'}-2} (Q_p \phi)(s) ds \right] \phi(t) dt \\
&\leq \int_{I_b} f(t) \left( \int_0^t \phi \right)^{\frac{1}{p'}} \left[ \int_{I_b} g \left[ \int_{I_b} (Q_p \phi)(t) dt \right]^{-\frac{1}{p'}-1} \right. \\
&\quad \left. + \left( \frac{1}{p'} + 1 \right) \int_t^b \int_0^s g [s (PQ_p)(\phi)(s)]^{-\frac{1}{p'}-2} (Q_p \phi)(s) ds \right] \phi(t) dt \\
&= \int_{I_b} f(t) \left( \int_0^t \phi \right)^{\frac{1}{p'}} \left[ p^{\frac{1}{p'}+1} \int_{I_b} g \left[ \int_{I_b} \phi \right]^{-\frac{1}{p'}-1} \right. \\
&\quad \left. + \left( \frac{1}{p'} + 1 \right) \int_t^b \int_0^s g [s (PQ_p)(\phi)(s)]^{-\frac{1}{p'}-2} (Q_p \phi)(s) ds \right] \phi(t) dt \\
&\leq (p+1)^2 \left[ \int_{I_b} f^p \phi \right]^{\frac{1}{p}} \left[ \int_{I_b} \int_0^t \phi \left[ \int_{I_b} g \left[ \int_{I_b} \phi \right]^{-\frac{1}{p'}-1} \right. \right. \\
&\quad \left. \left. + \int_t^b \int_0^s g [s (PQ_p)(\phi)(s)]^{-\frac{1}{p'}-2} (Q_p \phi)(s) ds \right]^{p'} \phi(t) dt \right]^{\frac{1}{p'}} \\
&\leq (p+1)^2 \left[ \int_{I_b} f^p \phi \right]^{\frac{1}{p}} \left[ \int_{I_b} g \left[ \int_{I_b} \phi \right]^{-\frac{1}{p'}-1} \left[ \int_{I_b} \phi(t) \int_0^t \phi dt \right]^{\frac{1}{p'}} \right. \\
&\quad \left. + \left[ \int_{I_b} \left( \int_t^b \int_0^s g [s (PQ_p)(\phi)(s)]^{-\frac{1}{p'}-2} (Q_p \phi)(s) ds \right)^{p'} \phi(t) \int_0^t \phi dt \right]^{\frac{1}{p'}} \right] \\
&\leq (p+1)^2 \left[ \int_{I_b} f^p \phi \right]^{\frac{1}{p}} \left[ \int_{I_b} g \left[ \int_{I_b} \phi \right]^{-\frac{1}{p}} \right. \\
&\quad \left. + \left[ \int_{I_b} \left( \int_t^b \int_0^s g [s (PQ_p)(\phi)(s)]^{-\frac{1}{p'}-2} (Q_p \phi)(s) ds \right)^{p'} \phi(t) \int_0^t \phi dt \right]^{\frac{1}{p'}} \right],
\end{aligned}$$

in which the third inequality was obtained using Hölder's inequality with respect to the measure  $\phi(t)dt$ .

The proof of (i) will be complete if we can show, that

$$\int_{I_b} \left( \int_t^b \int_0^s g[s(PQ_p)(\phi)(s)]^{-\frac{1}{p'}-2} (Q_p \phi)(s) ds \right)^{p'} \phi(t) \int_0^t \phi dt$$

is dominated by a constant multiple of  $\int_{I_b} (Pg)(t)^{p'} \psi(t) dt$ . To this end, let

$$H(t) := \int_0^s g[s(PQ_p)(\phi)(s)]^{-\frac{1}{p'}-2} (Q_p \phi)(s), \quad s \in I_b$$

so that the assertion reads

$$\begin{aligned} \int_{I_b} \left( \int_t^b H(s) ds \right)^{p'} \phi(t) \int_0^t \phi dt \\ \leq C \int_{I_b} H(t)^{p'} [t(PQ_p)(\phi)(t)]^{p'} ((Q_p \phi)(t))^{-p'-1} \int_0^t \phi(t) dt \end{aligned}$$

But, this holds by Theorem 3.1, inasmuch as

$$\begin{aligned} & \left( \int_0^t \phi(s) \int_0^s \phi ds \right)^{\frac{1}{p'}} \left( \int_t^b [s(PQ_p)(\phi)(s)]^{-p} (Q_p \phi)(s) \left( \int_0^s \phi \right)^{1-p} ds \right)^{\frac{1}{p}} \\ &= 2^{-\frac{1}{p'}} \left( \int_0^t \phi \right)^{\frac{2}{p'}} \left( \int_t^b [s(PQ_p)(\phi)(s)]^{-p} (Q_p \phi)(s) \left( \int_0^s \phi \right)^{1-p} ds \right)^{\frac{1}{p}} \\ &\leq 2^{-\frac{1}{p'}} \left( \int_0^t \phi \right)^{\frac{2}{p'}} \left( \int_0^t \phi \right)^{-\frac{1}{p'}} \left( \int_t^b [s(PQ_p)(\phi)(s)]^{-p} d[s(PQ_p)(\phi)(s)] \right)^{\frac{1}{p}} \\ &= -\frac{2^{-\frac{1}{p'}}}{p-1} \left( \int_0^t \phi \right)^{\frac{1}{p'}} \left( [s(PQ_p)(\phi)(s)]^{-p+1} \Big|_t^b \right)^{\frac{1}{p}} \\ &\leq \frac{2^{-\frac{1}{p'}}}{p-1} \left[ \frac{\int_0^t \phi}{t(PQ_p)(\phi)(t)} \right]^{\frac{1}{p'}} \\ &= \frac{2^{-\frac{1}{p'}}}{p-1} \left[ \frac{\int_0^t \phi}{\int_0^t \phi + t(Q_p \phi)(t)} \right]^{\frac{1}{p'}} \\ &\leq \frac{2^{-\frac{1}{p'}}}{p-1}. \end{aligned}$$

(ii) To begin, suppose  $b = \infty$ . Making the change of variable  $t \rightarrow t^{-1}$  three times in a row and setting  $\tilde{f}(y) = f(y^{-1})$ ,  $\tilde{g}(y) = g(y^{-1})$ ,  $\tilde{\phi}(y) = \phi(y^{-1})y^{p-2}$ , we obtain

$$\int_0^\infty fg \left[ \frac{Q_p \phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p'}+1} = \int_0^\infty \tilde{f}\tilde{g} \left[ \frac{\int_{t^{-1}}^\infty \phi(s)s^{-p}ds}{t^p \int_0^{t^{-1}} Q_p \phi} \right]^{\frac{1}{p'}+1} dt,$$



$$\begin{aligned}
&= \int_0^\infty \tilde{f} \tilde{g} \left[ \frac{\int_0^t \tilde{\phi}}{t^p \int_t^\infty s^{-p-1} \int_0^s \tilde{\phi} ds} \right]^{\frac{1}{p'}+1} dt \\
&= \int_0^\infty \tilde{f} \tilde{g} \left[ \frac{\int_0^t \tilde{\phi}}{t (Q_p P) (\tilde{\phi})(t)} \right]^{\frac{1}{p'}+1} dt \\
&= \int_0^\infty \tilde{f} \tilde{g} \left[ \frac{P \tilde{\phi}}{(P Q_p) (\tilde{\phi})} \right]^{\frac{1}{p'}+1}.
\end{aligned}$$

Thus, from (i), there follows, since  $\tilde{f} \downarrow$ ,

$$\int_0^\infty f g \left[ \frac{Q_p \phi}{(P Q_p) (\phi)} \right]^{\frac{1}{p'}+1} \leq C \left[ \int_0^\infty \tilde{f}^p \tilde{\phi} \right]^{\frac{1}{p}} \left[ \left( \int_0^\infty (P \tilde{g})^{p'} \tilde{\psi} \right)^{\frac{1}{p'}} + \frac{\int_0^\infty \tilde{g}}{\left[ \int_0^\infty \tilde{\phi} \right]^{\frac{1}{p}}} \right],$$

with

$$\tilde{\psi} = \frac{(P \tilde{\phi})(Q_p \tilde{\phi})}{\left[ (P Q_p) (\tilde{\phi}) \right]^{p'+1}}.$$

Now, the change of variable  $t \rightarrow t^{-1}$  yields

$$\begin{aligned}
\int_0^\infty \tilde{f}(t) \tilde{\phi}(t) dt &= \int_0^\infty \tilde{f}(t^{-1})^p \tilde{\phi}(t^{-1}) t^{-2} dt \\
&= \int_0^\infty f(t)^p \phi(t) t^{-p} dt, \\
\int_0^\infty \tilde{g}(t) dt &= \int_0^\infty \tilde{g}(t^{-1}) t^{-2} dt = \int_0^\infty g(t) dt
\end{aligned}$$

and

$$\int_0^\infty \tilde{\phi}(t) dt = \int_0^\infty \tilde{\phi}(t^{-1}) t^{-2} dt = \int_0^\infty \phi(t) t^{-p} dt = \infty$$

Again,

$$\begin{aligned}
\tilde{\psi}(t) &= \frac{t^{-1} \int_0^t \tilde{\phi}(s) ds t^{p-1} \int_t^\infty \tilde{\phi}(s) s^{-p} ds}{\left[ (P Q_p) (\tilde{\phi})(t) \right]^{p'+1}} \\
&= \frac{t^{-1} \int_0^t \phi(s^{-1}) s^{p-2} ds t^{p-1} \int_t^\infty \phi(s^{-1}) s^{-2} ds}{\left[ \frac{t^{-1}}{p} \int_0^t \phi(s^{-1}) s^{p-2} ds + \frac{t^{p-1}}{p} \int_t^\infty \phi(s^{-1}) s^{-2} ds \right]^{p'+1}} \\
&= \frac{t^{-1} \int_0^{t^{-1}} \phi(s) ds t^{p-1} \int_{t^{-1}}^\infty \phi(s) s^{-p} ds}{\left[ \frac{t^{-1}}{p} \int_{t^{-1}}^\infty \phi(s) s^{-p} ds + \frac{t^{p-1}}{p} \int_0^{t^{-1}} \phi(s) ds \right]^{p'+1}}
\end{aligned}$$

$$\begin{aligned}
&= t^{p'-2} \frac{(P\phi)(t^{-1})(Q_p\phi)(t^{-1})}{[(PQ_p)(\phi)(t^{-1})]^{p'+1}} \\
&= \psi(t^{-1})t^{p'-2}.
\end{aligned}$$

So,

$$\begin{aligned}
\int_0^\infty (P\tilde{g})(t)^p \tilde{\psi}(t) dt &= \int_0^\infty \left( t^{-1} \int_0^t g(s^{-1})s^{-2} ds \right)^{p'} \psi(t^{-1})t^{p'-2} dt \\
&= \int_0^\infty \left( \int_0^t g(s^{-1})s^{-2} ds \right)^{p'} \psi(t^{-1})t^{-2} dt \\
&= \int_0^\infty \left( \int_t^\infty g(s) ds \right)^{p'} \psi(t) dt.
\end{aligned}$$

This completes the proof of (ii) when  $b = \infty$ . In the case  $b < \infty$ , a similar argument works if we replace the transformation  $t \rightarrow t^{-1}$  by  $t \rightarrow (b - t)^{-1}$ .  $\square$

**Proof of Theorem A.** We first show

$$(4.2) \quad \rho'_{p,\phi}(g) \geq c \left[ \rho_{p',\psi}(g) + \frac{\int_{I_b} |g|}{\left[ \int_{I_b} \phi \right]^{\frac{1}{p}}} \right],$$

for some  $c > 0$  independent of  $g \in \mathfrak{M}_+(I_b)$ . To this end, it suffices, in view of (2.2), to find constants  $C, c > 0$ , independent of  $g \in \rho'_{p,\phi}$ , to which there corresponds an  $f \in \mathfrak{M}_+(I_b)$ , with  $f \downarrow$ ,  $\rho_{p,\phi}(f) \leq C$  and

$$(4.3) \quad \int_{I_b} f g^* \geq c \left[ \rho_{p',\psi}(g^*) + \frac{\int_{I_b} g^*}{\left[ \int_{I_b} \phi \right]^{\frac{1}{p}}} \right].$$

Fixing  $g$ , we seek

$$f = Qh$$

for some  $h$  in  $\mathfrak{M}_+(I_b)$ .

We need a condition on  $h$  to guarantee  $\rho_{p,\phi}(Qh) < \infty$ . But,

$$\begin{aligned}
\rho_{p,\phi}(Qh) &= \left[ \int_{I_b} ((PQ)h)^p \phi \right]^{\frac{1}{p}} \\
&= \left[ \int_{I_b} (Ph + Qh)^p \phi \right]^{\frac{1}{p}} \\
&\leq 2 \left[ \int_{I_b} (Sh)^p \phi \right]^{\frac{1}{p}} \\
&\leq B \left[ \int_{I_b} h^p \psi^{1-p} \right]^{\frac{1}{p}},
\end{aligned}$$

the last inequality being proved in the Appendix. The desired condition on  $h$  is thus

$$\int_{I_b} h^p \psi^{1-p} < \infty.$$

As pointed out in Section 3, the weighted Lebesgue norms

$$\left[ \int_{I_b} g^{p'} \psi \right]^{\frac{1}{p'}} \quad \text{and} \quad \left[ \int_{I_b} h^p \psi^{1-p} \right]^{\frac{1}{p}}, \quad g, h \in \mathfrak{M}_+(I_b),$$

are dual to one another. Therefore, for our given  $g \in L_{\rho'_{p,\phi}}$ , there exists  $h_0 \in \mathfrak{M}_+(I_b)$ , such that

$$\int_{I_b} h_0^p \psi^{1-p} \leq 1$$

and

$$\int_{I_b} g^* Q h_0 = \int_{I_b} h_0 P g^* \geq \frac{1}{2} \left[ \int_{I_b} (g^{**})^{p'} \psi \right]^{\frac{1}{p'}} = \frac{1}{2} \rho_{p',\psi}(g).$$

If  $\int_{I_b} \phi < \infty$ , the constant function with value

$$\frac{1}{\left[ \int_{I_b} \phi \right]^{\frac{1}{p}}}$$

will belong to  $\Gamma_{p,\phi}$  with norm 1 and

$$(4.4) \quad f := Q h_0 + \frac{1}{\left[ \int_{I_b} \phi \right]^{\frac{1}{p}}}$$

will satisfy

$$\int_{I_b} f g^* \geq \frac{\int_{I_b} g^*}{\left[ \int_{I_b} \phi \right]^{\frac{1}{p}}}.$$

Altogether, then, the function  $f$  defined in (4.4) has  $\rho_{p,\phi}(f) \leq C = B + 1$  and satisfies (4.3) with  $c = \frac{1}{2}$ .

We now prove the inequality opposite to (4.2), this being equivalent to

$$(4.5) \quad \int_{I_b} f^* g^* \leq C \rho_{p,\phi}(f^*) \left[ \rho_{p',\psi}(g^*) + \frac{\int_{I_b} g^*}{\left[ \int_{I_b} \phi \right]^{\frac{1}{p}}} \right],$$

in which  $C > 0$  is independent of  $f, g \in \mathfrak{M}(X)$ .

It suffices to consider  $g^*$  of the form

$$g^* = k + Qh, \quad k \geq 0 \quad \text{and} \quad h \in \mathfrak{M}_+(I_b).$$

For the term

$$\frac{\int_{I_b} g^*}{\left[\int_{I_b} \phi\right]^{\frac{1}{p}}}$$

to be finite we require  $b = \mu(X) < \infty$  or  $\int_{I_b} \phi = \infty$ . In either case, the term is dominated by an absolute constant times  $\rho_{p',\psi}(g^*)$  and is irrelevant.

We have only to consider those  $g^*$  of the form  $g^* = Qh$ ,  $h \in \mathfrak{M}_+(I_b)$ . For such  $g^*$ ,

$$\begin{aligned} \int_{I_b} f^* g^* &= \int_{I_b} f^* Qh = \int_{I_b} h P(f^*) = \int_{I_b} f^{**} h \\ &= \int_{I_b} f^{**} h [(PQ_p)(\phi)]^{\frac{1}{p'}+1} [(PQ_p)(\phi)]^{-\frac{1}{p'}-1} \\ &= p^{-\frac{1}{p'}-1} \int_{I_b} f^{**} h \left[ \frac{P\phi + Q_p\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p'}+1} \\ (4.6) \quad &\leq \left(\frac{2}{p}\right)^{\frac{1}{p'}+1} \left[ \int_{I_b} f^{**} h \left[ \frac{P\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p'}+1} + \int_{I_b} f^{**} h \left[ \frac{Q_p\phi}{(PQ_p)(\phi)} \right]^{\frac{1}{p'}+1} \right] \\ &= \left(\frac{2}{p}\right)^{\frac{1}{p'}+1} [I_1 + I_2]. \end{aligned}$$

Since  $f^{**} \downarrow$ , Lemma 4.1, (i), gives

$$(4.7) \quad I_1 \leq C \rho_{p,\phi}(f) \left[ \left( \int_{I_b} (Ph)^{p'} \psi \right)^{\frac{1}{p'}} + \frac{\int_{I_b} h}{\left[\int_{I_b} \phi\right]^{\frac{1}{p}}} \right].$$

But,

$$g^{**} = Pg^* = (PQ)h = Ph + Qh \geq Ph$$

and

$$\int_{I_b} g^* = \int_{I_b} Qh = \int_{I_b} h,$$

whence (4.7) implies

$$(4.8) \quad I_1 \leq C \rho_{p,\phi}(f) \left[ \rho_{p',\psi}(g^*) + \frac{\int_{I_b} g^*}{\left[\int_{I_b} \phi\right]^{\frac{1}{p}}} \right].$$

Observing that  $\int_0^t f^* \uparrow$ , Lemma 4.1, (ii), ensures

$$(4.9) \quad I_2 \leq C \left[ \int_{I_b} \left( \int_0^t f^* \right)^p \phi(t) t^{-p} dt \right]^{\frac{1}{p}} \left( \int_{I_b} (Qh)^{p'} \psi \right)^{\frac{1}{p'}} = C \rho_{p,\phi}(f^*) \rho_{p',\psi}(g^*)$$

Combining (4.6), (4.8) and (4.9) yields (4.5) and thereby completes the proof.  $\square$

**Corollary 4.2.** *Let  $\phi$  be a non-trivial weight function on  $\mathbb{R}_+$ , and  $\psi$  its dual weight. Then,*

$$(4.10) \quad t^{-p'} \left[ \int_0^t \psi(s) ds + t^{p'} \int_t^\infty \psi(s) s^{-p'} ds \right] \approx \left[ \int_0^t \phi(s) ds + t^p \int_t^\infty \phi(s) s^{-p} ds \right]^{1-p'}.$$

*Proof.* It is easy to see that

$$\rho_{p,\phi}(\chi_{(0,t)}) = \left( \int_0^t \phi(s) ds + t^p \int_t^\infty \phi(s) s^{-p} ds \right)^{\frac{1}{p}}$$

and

$$\rho_{p',\psi}(\chi_{(0,t)}) = \left( \int_0^t \psi(s) ds + t^{p'} \int_t^\infty \psi(s) s^{-p'} ds \right)^{\frac{1}{p'}}.$$

Since  $\rho_{p,\phi}$  and  $\rho_{p',\psi}$  are associate r.i. function norms, (4.10) now follows from (2.3).  $\square$

**Corollary 4.3.** *Fix  $p \in (1, \infty)$  and suppose  $\phi$  is a non-trivial weight function on  $\mathbb{R}_+$ , with*

$$\int_0^\infty \phi(t) dt = \infty.$$

*Then,*

$$(4.11) \quad \sup_{f \in \Omega_{0,1}(\mathbb{R}_+)} \frac{\int_0^\infty fg}{\left( \int_0^\infty f^p \phi \right)^{\frac{1}{p}}} \approx \left( \int_0^\infty (Sg)^{p'} \psi \right)^{\frac{1}{p'}}, \quad g \in \mathfrak{M}_+(\mathbb{R}_+),$$

*in which  $\psi$  is the weight dual to  $\phi$  and*

$$\Omega_{0,1}(\mathbb{R}_+) := \{f \in \mathfrak{M}_+(\mathbb{R}_+) : tf(t) \uparrow \text{ and } f \downarrow\}$$

*Proof.* As pointed out in [2, p. 117],  $f \in \Omega_{0,1}(\mathbb{R}_+)$  if and only if

$$\frac{1}{2} t^{-1} \int_0^t h^*(s) ds \leq f(t) \leq 2t^{-1} \int_0^t h^*(s) ds,$$

for some  $h \in \mathfrak{M}_+(\mathbb{R}_+)$ . Hence, the left side of (4.11), is equivalent to

$$\begin{aligned} \sup_{h \in \mathfrak{M}_+(\mathbb{R}_+)} \frac{\int_0^\infty t^{-1} \int_0^t h^*(s) ds g(t) dt}{\rho_{p,\phi}(h)} &= \sup_{h \in \mathfrak{M}_+(\mathbb{R}_+)} \frac{\int_0^\infty h^*(t) \int_t^\infty g(s) \frac{ds}{s} dt}{\rho_{p,\phi}(h)} \\ &= \rho'_{p,\phi} \left( \int_t^\infty g(s) \frac{ds}{s} \right) \\ &\approx \rho_{p',\psi} \left( \int_t^\infty g(s) \frac{ds}{s} \right), \end{aligned}$$

which yields (4.11), in view of (3.3), since

$$\rho_{p',\psi} \left( \int_t^\infty g(s) \frac{ds}{s} \right) = \left( \int_0^\infty \left( t^{-1} \int_0^t \int_s^\infty g(y) \frac{dy}{y} ds \right)^{p'} \psi(t) dt \right)^{\frac{1}{p'}}$$

$$\approx \left( \int_0^\infty (Sg)^{p'} \psi \right)^{\frac{1}{p'}}.$$

□

**Theorem 4.4.** Fix  $p, q \in (1, \infty)$ . Suppose  $\phi_1$  and  $\phi_2$  are weights on  $\mathbb{R}_+$ , with  $\phi_1$  and its dual weight  $\psi_1$  as in Corollary 4.3. Let  $T$  be a positive linear operator on  $\mathfrak{M}_+(\mathbb{R}_+)$  with associate operator  $T'$ . Then,

$$(4.12) \quad \left( \int_0^\infty (Tf)^q \phi_2 \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f^p \phi_1 \right)^{\frac{1}{p}}, \quad f \in \Omega_{0,1}(\mathbb{R}_+),$$

if and only if

$$(4.13) \quad \left( \int_0^\infty (ST')(h)^{p'} \psi_1 \right)^{\frac{1}{p'}} \leq K \left( \int_0^\infty h^{q'} \phi_2^{1-q'} \right)^{\frac{1}{q'}}, \quad h \in \mathfrak{M}_+(\mathbb{R}_+),$$

or

$$(4.14) \quad \left( \int_0^\infty (TS)(h)^q \phi_2 \right)^{\frac{1}{q}} \leq K \left( \int_0^\infty h^p \psi_1^{1-p} \right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}_+(\mathbb{R}_+).$$

Here,  $K \approx C$ .

*Proof.* The reverse Hölder inequality ensures that (4.12) is equivalent to

$$\frac{\int_0^\infty (Tf)h}{\left( \int_0^\infty h^{q'} \phi_2^{1-q'} \right)^{\frac{1}{q'}}} \lesssim \left( \int_0^\infty f^p \phi_1 \right)^{\frac{1}{p}}, \quad f \in \Omega_{0,1}(\mathbb{R}_+), \quad h \in \mathfrak{M}_+(\mathbb{R}_+),$$

or

$$(4.15) \quad \frac{\int_0^\infty f(T'h)}{\left( \int_0^\infty f^p \phi_1 \right)^{\frac{1}{p}}} \lesssim \left( \int_0^\infty h^{q'} \phi_2^{1-q'} \right)^{\frac{1}{q'}}, \quad f \in \Omega_{0,1}(\mathbb{R}_+), \quad h \in \mathfrak{M}_+(\mathbb{R}_+).$$

In view of Corollary 4.3, (4.15) amounts to

$$\rho_{p', \psi_1}((ST')(h)) \lesssim \left( \int_0^\infty h^{q'} \phi_2^{1-q'} \right)^{\frac{1}{q'}}, \quad h \in \mathfrak{M}_+(\mathbb{R}_+),$$

that is, (4.13). As we have

$$\int_0^\infty (ST')h(t)g(t)dt = \int_0^\infty h(t)(TS)g(t)dt, \quad h, g \in \mathfrak{M}_+(\mathbb{R}_+),$$

(4.13) is equivalent to (4.14), by the duality theorem for weighted Lebesgue spaces

□

## 5. IMBEDDINGS AND BOYD INDICES

**Theorem 5.1.** *Fix  $p, q \in (1, \infty)$ . Suppose  $\phi_1$  and  $\phi_2$  are weights on  $\mathbb{R}_+$ , with  $\phi_1$  and its dual weight  $\psi_1$  as in Corollary 4.3. Then, the (possibly infinite) norm of the imbedding*

$$(5.1) \quad \Gamma_{p,\phi_1}(\mathbb{R}_+) \hookrightarrow \Gamma_{q,\phi_2}(\mathbb{R}_+)$$

is equivalent to

$$(5.2) \quad \sup_{t>0} \frac{\left[ \int_0^t \phi_2(s) ds + t^q \int_t^\infty \phi_2(s) s^{-q} ds \right]^{\frac{1}{q}}}{\left[ \int_0^t \phi_1(s) ds + t^p \int_t^\infty \phi_1(s) s^{-p} ds \right]^{\frac{1}{p}}},$$

if  $1 < p \leq q < \infty$ , and to

$$(5.3) \quad \left[ \int_0^\infty \left[ \frac{\int_0^t \phi_2(s) ds + t^q \int_t^\infty \phi_2(s) s^{-q} ds}{\int_0^t \phi_1(s) ds + t^p \int_t^\infty \phi_1(s) s^{-p} ds} \right]^{\frac{q}{p-q}} \phi_2(t) dt \right]^{\frac{1}{q} - \frac{1}{p}},$$

if  $1 < q < p < \infty$ .

*Proof.* The imbedding (5.1) is equivalent to an inequality of the form

$$(5.4) \quad \rho_{q,\phi_2}(If) \leq C \rho_{p,\phi_1}(f), \quad f \in \Omega_{0,1}(\mathbb{R}_+),$$

or

$$(5.5) \quad \left( \int_0^\infty (If)^q \phi_2 \right)^{\frac{1}{q}} \leq C \left( \int_0^\infty f^p \phi_1 \right)^{\frac{1}{p}}, \quad f \in \Omega_{0,1}(\mathbb{R}_+),$$

in which  $I$  is the identity operator. According to Theorem 4.4, (5.5) reduces to

$$(5.6) \quad \left( \int_0^\infty (Sh)^q \phi_2 \right)^{\frac{1}{q}} \leq K \left( \int_0^\infty h^p \psi_1^{1-p} \right)^{\frac{1}{p}}, \quad h \in \mathfrak{M}_+(\mathbb{R}_+);$$

here,  $K \approx C$  and

$$\psi_1(t) = \frac{(P\phi_1)(t)(Q_p\phi_1)(t)}{[(PQ_p)(\phi_1)]^{p'+1}}.$$

By Theorem 3.2, the least possible  $K$  in (5.6) is equivalent to

$$(5.7) \quad \sup_{t>0} \left[ \int_0^\infty \frac{\psi_1(s)}{(s+t)^{p'}} ds \right]^{\frac{1}{p'}} \left[ \int_0^\infty \left( \frac{t}{s+t} \right)^q \phi_2(s) ds \right]^{\frac{1}{q}}$$

when  $1 < p \leq q < \infty$ , and to

$$(5.8) \quad \left[ \int_0^\infty \left[ \int_0^\infty \frac{\psi_1(s)}{(s+t)^{p'}} ds \right]^{\frac{(p-1)q}{p-q}} \left[ \int_0^\infty \left( \frac{t}{s+t} \right)^q \phi_2(s) ds \right]^{\frac{q}{p-q}} \phi_2(t) dt \right]^{\frac{1}{q} - \frac{1}{p}},$$

when  $1 < q < p < \infty$ . But,

$$\int_0^\infty \left( \frac{t}{s+t} \right)^q \phi_2(s) ds \approx \int_0^t \phi_2(s) ds + t^q \int_t^\infty \phi_2(s) s^{-q} ds$$

and

$$\begin{aligned} \int_0^\infty \frac{\psi_1(s)}{(s+t)^{p'}} ds &\approx t^{-p'} \int_0^t \psi_1(s) ds + \int_t^\infty \psi_1(s) s^{-p'} ds \\ &= t^{-p'} \left[ \int_0^t \psi_1(s) ds + t^{p'} \int_t^\infty \psi_1(s) s^{-p'} ds \right] \\ &\approx \left[ \int_0^t \phi_1(s) ds + t^p \int_t^\infty \phi_1(s) s^{-p} ds \right]^{1-p'}, \end{aligned}$$

by Corollary 4.2, so (5.7) becomes (5.2) and (5.8) becomes (5.3).  $\square$

**Theorem 5.2.** Fix an index  $p$ ,  $1 < p < \infty$  and suppose  $\phi$  is a non-trivial weight on  $\mathbb{R}_+$ . Take  $\rho = \rho_{p,\phi}$  on  $\mathfrak{M}_+(\mathbb{R}_+)$ . Then,

$$(5.9) \quad h_\rho(t) \approx M_\rho(t) \approx \sup_{s \in \mathbb{R}_+} \left[ \frac{\int_0^{st} \phi(y) dy + s^{pt} \int_{st}^\infty \phi(y) y^{-p} dy}{\int_0^s \phi(y) dy + s^p \int_s^\infty \phi(y) y^{-p} dy} \right]^{\frac{1}{p}}, \quad t \in \mathbb{R}_+,$$

and

$$(5.10) \quad i_\rho = \underline{i}_\rho, \quad I_\rho = \underline{I}_\rho.$$

*Proof.* For  $f \in \mathfrak{M}_+(\mathbb{R}_+)$ ,  $f$  decreasing, we have

$$\left( E_{\frac{1}{t}} f \right)^{**} (s) = f^{**} \left( \frac{s}{t} \right), \quad s \in \mathbb{R}_+,$$

so

$$\rho_{p,\phi} \left( E_{\frac{1}{t}} f \right) = \rho_{p,\bar{\phi}}(f),$$

where

$$\bar{\phi}(s) = t\phi(st).$$

Thus, for  $t \in \mathbb{R}_+$ ,

$$\begin{aligned} h_\rho(t) &= \sup_{\substack{f \in \mathfrak{M}_+(\mathbb{R}_+) \\ f \downarrow}} \frac{\rho_{p,\phi} \left( E_{\frac{1}{t}} f \right)}{\rho_{p,\phi}(f)} \\ &= \sup_{f \in \mathfrak{M}_+(\mathbb{R}_+)} \frac{\rho_{p,\bar{\phi}}(f)}{\rho_{p,\phi}(f)} \\ &\approx \sup_{s \in \mathbb{R}_+} \frac{\rho_{p,\bar{\phi}} \left( \chi_{(0,s)} \right)}{\rho_{p,\phi} \left( \chi_{(0,s)} \right)}, \quad \text{by Theorem 5.1,} \\ &\approx \sup_{s \in \mathbb{R}_+} \left[ \frac{\int_0^s t\phi(ty) dy + s^p \int_s^\infty t\phi(ty) y^{-p} dy}{\int_0^s \phi(y) dy + s^p \int_s^\infty \phi(y) y^{-p} dy} \right]^{\frac{1}{p}} \end{aligned}$$



$$\begin{aligned}
&\approx \sup_{s \in \mathbb{R}_+} \left[ \frac{\int_0^{st} \phi(y) dy + s^p t^p \int_{st}^\infty \phi(y) y^{-p} dy}{\int_0^s \phi(y) dy + s^p \int_s^\infty \phi(y) y^{-p} dy} \right]^{\frac{1}{p}} \\
&\approx \sup_{s \in \mathbb{R}_+} \frac{\rho_{p,\phi}(\chi_{(0,st)})}{\rho_{p,\phi}(\chi_{(0,s)})} \\
&= M_\rho(t).
\end{aligned}$$

□

**Remark 5.3.** The formula (5.9), now proved, is the one asserted in Theorem B.

## 6. CALDERÓN-ZYGMUND OPERATORS

A function  $K$ , on  $\mathbb{R}^n \setminus \{0\}$ , locally integrable away from the origin, is said to be a Calderón-Zygmund (CZ) kernel, provided it satisfies the following four conditions:

(i) There exists a constant  $C_1 > 0$ , independent of  $\varepsilon$  and  $N$ ,  $0 < \varepsilon < N$ , such that

$$\left| \int_{\varepsilon < |x| < N} K(x) dx \right| \leq C_1;$$

moreover, for each  $N > 0$ , one has the existence of

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon < |x| < N} K(x) dx.$$

(ii) There exists a constant  $C_2 > 0$ , independent of  $R > 0$ , for which

$$\int_{|x| < R} |x| |K(x)| dx \leq C_2 R.$$

(iii) There exists a constant  $C_3 > 0$ , independent of  $y \in \mathbb{R}^n \setminus \{0\}$ , with

$$\int_{|x| > 2|y|} |K(x-y) - K(x)| dx \leq C_3.$$

(iv) There exists a constant  $C_4 > 0$ , independent of  $R > 0$  and of points  $x_1, x_2$  and  $x_3$  in  $\mathbb{R}^n$  within a distance  $\frac{R}{2}$  of one another and each a distance greater than  $R$  from  $y$ , such that

$$|K(x_1 - y) - K(x_2 - y)| \leq C_4 \frac{|x_1 - x_2|}{|x_3 - y|^{n+1}}.$$

The Calderón-Zygmund operator,  $T_K$ , with kernel  $K$ , is the singular integral operator

$$(T_K f)(x) := \lim_{\varepsilon \rightarrow 0^+} \int_{|x-y| > \varepsilon} K(x-y) f(y) dy, \quad x \in \mathbb{R}^n,$$

which is defined a.e. for all  $f \in \mathfrak{M}(\mathbb{R}^n)$  with

$$\int_{\mathbb{R}^n} \frac{|f(y)|}{1 + |y|^n} dy < \infty.$$

**Theorem 6.1.** Fix  $p$ ,  $1 < p < \infty$ , and suppose the weight  $\phi$  on  $\mathbb{R}_+$  satisfies

$$\int_0^\infty \phi(s) \min[1, s^{-p}] ds < \infty \quad \text{and} \quad \int_0^\infty \phi(s) \max[1, s^{-p}] ds = \infty.$$

Denote by  $\psi$  the function defined in (4.1).

Let  $T_K$  be a CZ operator. Then, one has

$$(6.1) \quad T_K : \Gamma_{p,\phi}(\mathbb{R}^n) \rightarrow \Gamma_{p,\phi}(\mathbb{R}^n)$$

if there exists  $c$ ,  $0 < c < 1$ , such that for all  $t \in \mathbb{R}_+$ ,

$$(6.2) \quad \begin{aligned} \int_0^{ct} \phi(s) ds + c^p t^p \int_t^\infty \phi(s) s^{-p} ds &\leq \frac{1}{2} \left[ \int_0^t \phi(s) ds + t^p \int_t^\infty \phi(s) s^{-p} ds \right] \\ \int_0^{ct} \psi(s) ds + c^{p'} t^{p'} \int_t^\infty \psi(s) s^{-p'} ds &\leq \frac{1}{2} \left[ \int_0^t \psi(s) ds + t^{p'} \int_t^\infty \psi(s) s^{-p'} ds \right]. \end{aligned}$$

*Proof.* Let  $\rho$  be an r.i. norm on  $\mathfrak{M}_+(\mathbb{R}_+)$  defined in terms of r.i. norm  $\bar{\rho}$  on  $\mathfrak{M}_+(\mathbb{R}_+)$  by  $\rho(f) = \bar{\rho}(f^*)$ . It is shown in [5] that

$$T_K : L_\rho(\mathbb{R}^n) \rightarrow L_\rho(\mathbb{R}^n)$$

provided

$$\lim_{s \rightarrow 0+} sh(s) = 0 = \lim_{s \rightarrow \infty} h(s),$$

where  $h(s) = h_{\bar{\rho}}(\frac{1}{s})$ . In terms of  $h_{\bar{\rho}}(s)$  and  $h_{\bar{\rho}'}(s) = sh(s)$ , these conditions read

$$(6.3) \quad \lim_{s \rightarrow 0+} h_{\bar{\rho}}(s) = 0 = \lim_{s \rightarrow 0+} h_{\bar{\rho}'}(s).$$

The inequalities

$$h_{\bar{\rho}}(s_1 s_2) \leq h_{\bar{\rho}}(s_1) h_{\bar{\rho}}(s_2) \quad \text{and} \quad h_{\bar{\rho}'}(s_1 s_2) \leq h_{\bar{\rho}'}(s_1) h_{\bar{\rho}'}(s_2), \quad s_1, s_2 \in \mathbb{R}_+,$$

imply that, given  $\varepsilon > 0$ , (6.3) is equivalent to the existence of  $c$ ,  $0 < c < 1$ , for which  $h_{\bar{\rho}}(c) < \varepsilon$  and  $h_{\bar{\rho}'}(c) < \varepsilon$ . By Theorem B, then, (6.3) is equivalent to (6.2), when  $\bar{\rho} = \rho_{p,\phi}$ .  $\square$

**Remark 6.2.** The condition (6.2) is also necessary for (6.1) when, for example,  $T_K$  is the Hilbert transform or one of the Riesz transforms.

## 7. APPENDIX

It is our purpose here to give an heuristic argument to motivate the choice of  $\psi$  in (4.1) when  $\phi$  is a non-trivial weight on  $I_b$  satisfying  $\int_{I_b} \phi = \infty$ .

Now,

$$\rho'_{p,\phi}(g) = \sup_{f \in \mathfrak{M}_+(I_b)} \frac{\int_0^b f^*(t) g^*(t) dt}{\left[ \int_0^b f^{**}(t)^p \phi(t) dt \right]^{\frac{1}{p}}} =: I(g), \quad g \in \mathfrak{M}_+(I_b).$$

It suffices to consider  $f(t) = \int_t^b h(s) \frac{ds}{s}$  for some  $h \in \mathfrak{M}_+(I_b)$ ,  $h \neq 0$  a.e.. Since, in that case,

$$\int_0^b f^*(t) g^*(t) = \int_0^b \int_t^b h(s) \frac{ds}{s} g^*(t) dt = \int_0^b h(t) g^{**}(t) dt$$

and

$$f^{**}(t) = t^{-1} \int_0^t \int_s^b h(y) \frac{dy}{y} ds = t^{-1} \int_0^t h(s) ds + \int_t^b h(s) \frac{ds}{s} \approx (Sh)(t), \quad t \in I_b,$$

we have

$$I(g) = \sup_{h \in \mathfrak{M}_+(I_b)} \frac{\int_0^b h(t) g^{**}(t) dt}{\left[ \int_0^b (Sh)(t)^p \phi(t) dt \right]^{\frac{1}{p}}}.$$

If  $\bar{\phi}$  is such that

$$(7.1) \quad \int_0^b (Sh)^p \phi \leq C \int_0^b h^p \bar{\phi}, \quad h \in \mathfrak{M}(I_b),$$

then,

$$I(g) \geq C^{-1} \sup_{h \in \mathfrak{M}_+(I_b)} \frac{\int_0^b h(t) g^{**}(t) dt}{\left[ \int_0^b h(t)^p \bar{\phi}(t) dt \right]^{\frac{1}{p}}}.$$

This suggests we take  $\psi(t) = \bar{\phi}(t)^{1-p'}$  where  $\bar{\phi}$  is, in some sense the smallest weight such that (7.1) holds. Andersen's condition (3.5) for (7.1) leads us to solve for  $\bar{\phi}(t)^{1-p'}$  in the equation

$$(7.2) \quad \left[ \int_0^b \frac{\phi(s)}{(s+t)^p} ds \right]^{\frac{1}{p}} \left[ \int_0^b \left( \frac{t}{s+t} \right)^{p'} \bar{\phi}(s)^{1-p'} ds \right]^{\frac{1}{p'}} = 1,$$

or, what is equivalent,

$$\int_0^t \bar{\phi}(s)^{1-p'} ds + t^{p'} \int_t^b \bar{\phi}(s)^{1-p'} s^{-p'} ds = \left[ t^{-p} \int_0^t \phi(s) ds + \int_t^b \phi(s) s^{-p} ds \right]^{1-p'}.$$

Differentiation with respect to  $t$  yields

$$t^{p'-1} \int_t^b \bar{\phi}(s)^{1-p'} s^{-p'} ds = \int_0^t \phi(s) ds \left[ t^{-p} \int_0^t \phi(s) ds + \int_t^b \phi(s) s^{-p} ds \right]^{-p'}.$$

Differentiating again with respect to  $t$  we get

$$\bar{\phi}(t)^{1-p'} = \frac{pp' t^{pp'-1} \int_0^t \phi(s) ds \int_t^b \phi(s) s^{-p} ds}{\left[ \int_0^t \phi(s) ds + t^p \int_t^b \phi(s) s^{-p} ds \right]^{p'+1}} - \frac{t^{p'} \phi(t)}{\left[ \int_0^t \phi(s) ds + t^p \int_t^b \phi(s) s^{-p} ds \right]^{p'}}$$

It seems we essentially have

$$(7.3) \quad \overline{\phi}(t)^{1-p'} = \frac{(P\phi)(t)(Q_p\phi)(t)}{[(P\phi)(t) + (Q_p\phi)(t)]^{p'+1}}.$$

The weight  $\widehat{\phi}(t)$  given by

$$\widehat{\phi}^{1-p'}(t) = \frac{t^{p'}\phi(t)}{\left[\int_0^t \phi(s)ds + t^p \int_t^b \phi(s)s^{-p}ds\right]^{p'}} = \frac{\phi(t)}{[(P\phi)(t) + (Q_p\phi)(t)]^{p'}}$$

is readily shown to satisfy Andersen's condition (7.2) and, hence, so will

$$\frac{(P\phi)(t)(Q_p\phi)(t)}{[(P\phi)(t) + (Q_p\phi)(t)]^{p'+1}} = \overline{\phi}(t)^{1-p'} + \widehat{\phi}(t)^{1-p'}.$$

Now,  $\overline{\phi}(t)$  will be better than  $\widehat{\phi}(t)$  in (7.1) if

$$\int_0^b g^{**}(t)^{p'} \widehat{\phi}(t)^{1-p'} dt \leq C \int_0^b g^{**}(t)^{p'} \overline{\phi}(t)^{1-p'} dt.$$

One readily infers from Theorem 5.1 that this will be so if and only if

$$\int_0^t s^{p'-1} \int_s^b \widehat{\phi}(y)^{1-p'} y^{-p'} dy ds \leq C \int_0^t s^{p'-1} \int_s^b \overline{\phi}(y)^{1-p'} y^{-p'} dy ds.$$

But,

$$\begin{aligned} \int_s^b \overline{\phi}(y)^{1-p'} y^{-p'} dy &\approx \int_s^b y^{-p'} \frac{(P\phi)(y)(Q_p\phi)(y)}{[(P\phi)(y) + (Q_p\phi)(y)]^{p'+1}} dy \\ &= \int_s^b \frac{y^{p-1} \int_0^y \phi(z)dz \int_y^b \phi(z)z^{-p}dz}{\left[\int_0^y \phi(z)dz + y^p \int_y^b \phi(z)z^{-p}dz\right]^{p'+1}} dy \\ &= -\frac{1}{p'} \int_s^b \int_0^y \phi(z)dz \frac{d}{dz} \left[ \int_0^y \phi(z)dz + y^p \int_y^b \phi(z)z^{-p}dz \right]^{-p'} dy \\ &= -\frac{1}{p'} \int_0^y \phi(z)dz \left[ \int_0^y \phi(z)dz + y^p \int_y^b \phi(z)z^{-p}dz \right]^{-p'} \Big|_s^b \\ &\quad + \frac{1}{p'} \int_s^b \phi(y) \left[ \int_0^y \phi(z)dz + y^p \int_y^b \phi(z)z^{-p}dz \right]^{-p'} dy \\ &\geq \frac{1}{p'} \int_s^b \widehat{\phi}(y)^{1-p'} y^{-p'} dy, \end{aligned}$$

if  $\int_0^b \phi(z)dz = \infty$ .

## REFERENCES

- [1] K. F. Andersen, *Weighted inequalities for the Stieltjes transformation and Hilbert's double series*, Proc. Roy. Soc. Edinburgh Sect. A 86 (1980), no. 1-2, 7584.
- [2] J. Bergh and J. Lofstrom, *Interpolation Spaces. An Introduction*, Springer Verlag, New York 1976.
- [3] A. Gogatishvili and L. Pick, *Discretization and antidiscretization of rearrangement-invariant norms* Publ. Mat., 47 (2003), no. 2, 311 - 358. MR2006487 (2005f:46053)
- [4] M. L. Gol'dman, H. P. Heinig and V. D. Stepanov, *On the principle of duality in Lorentz spaces*, Canad. J. Math., 48 (1996), 959–979.
- [5] R. Kerman, *A sharp estimate for the least concave majorant and the range of Caldéron-Zygmund operators*, in preperation.
- [6] V.G. Maz'ya, Sobolev spaces, Springer-Verlang, Berlin, 1985.
- [7] G. Sinnamon, *A note on the Stieltjes transformation*, Proc. Roy. Soc. Edinburgh Sect. A 110 (1988), no. 1-2, 7378.
- [8] G. Sinnamon, *Embeddings of concave functions and duals of Lorentz spaces*, Publ. Mat. 46 (2002), no. 2, 489515.

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